DYNAMICS, EQUATIONS AND APPLICATIONS

BOOK OF ABSTRACTS
SESSION D21

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PLENARY LECTURES

GENERIC CONSERVATIVE DYNAMICS

Artur Avila
Universität Zürich, Switzerland & IMPA, Brazil

ON THE REGULARITY OF STABLE SOLUTIONS TO SEMILINEAR ELLIPTIC PDES

Alessio Figalli
ETH Zürich, Switzerland

Stable solutions to semilinear elliptic PDEs appear in several problems. It is known since the 1970's that, in dimension $n > 9$, there exist singular stable solutions. In this talk I will describe a recent work with Cabré, Ros-Oton, and Serra, where we prove that stable solutions in dimension $n \leq 9$ are smooth. This answers also a famous open problem, posed by Brezis, concerning the regularity of extremal solutions to the Gelfand problem.

RANDOM LOOPS

Martin Hairer
Imperial College London, UK

2D PERCOLATION REVISITED

Stanislav Smirnov
University of Geneva, Switzerland & Skoltech, Russia
Joint work with **Mikhail Khristoforov**.

We will discuss the state of our understanding of 2D percolation, and will present a recent joint work with Mikhail Khristoforov, giving a new proof of its conformal invariance at criticality.

STABILITY AND NONLINEAR PDES IN MIRROR SYMMETRY

Shing-Tung Yau
Harvard University, USA

I shall give a talk about a joint work that I did with Tristan Collins on an important nonlinear system equation of Monge-Ampère type. It is motivated from the theory of Mirror symmetry in string theory. I shall also talk about its algebraic geometric meaning.

FROM CLASSICAL TO QUANTUM AND BACK

Maciej Zworski
University of California, Berkeley, USA

Microlocal analysis exploits mathematical manifestations of the classical/quantum (particle/wave) correspondence and has been a successful tool in spectral theory and partial differential equations. We can say that these two fields lie on the "quantum/wave side".

In the last few years microlocal methods have been applied to the study of classical dynamical problems, in particular of chaotic flows. That followed the introduction of specially tailored spaces by Blank-Keller-Liverani, Baladi-Tsujii and other dynamicists and their microlocal interpretation by Faure-Sjoestrand and by Dyatlov and the speaker.

I will explain this microcar/dynamical connection in the context of Ruelle resonances, decay of correlations and meromorphy of dynamical zeta functions. I will also present some recent advances, among them results by Dyatlov-Guillarmou (Smale's conjecture on meromorphy of zeta functions for Axiom A flows), Guillarmou-Lefeuvres (local determination of metrics by the length spectrum) and Dang-Rivière (Ruelle resonances and Witten Laplacian).

PUBLIC LECTURE

FROM OPTIMAL TRANSPORT TO SOAP BUBBLES AND CLOUDS: A PERSONAL JOURNEY

Alessio Figalli
ETH Zürich, Switzerland

In this talk I'll give a general overview, accessible also to non-specialists, of the optimal transport problem. Then I'll show some applications of this theory to soap bubbles (isoperimetric inequalities) and clouds (semigeostrophic equations), problems on which I worked over the last 10 years. Finally, I will conclude with a brief description of some results that I recently obtained on the study of ice melting into water.

INVITED TALKS OF PART D2

DIFFERENTIABILITY OF THE FLOW FOR BV VECTOR FIELDS

Stefano Bianchini
SISSA, Italy

We show that the Regular Lagrangian Flow $X(t, y)$ generated by nearly incompressible BV vector fields admits a derivative $\nabla X(t, y)$ in the sense of measure. This matrix satisfies the ODE

$$\frac{d}{dt}\nabla X(t, y) = (D\mathbf{b}(t))_y \nabla X(t-, y)$$

where $(D\mathbf{b})_y$ is the disintegration of the measure $\int D\mathbf{b}(t)dt$ w.r.t. the trajectories $X(t, y)$.

ON THE LARGE TIME BEHAVIOR OF SOLUTIONS TO BIRTH AND SPREAD TYPE EQUATIONS

Yoshikazu Giga
University of Tokyo, Japan
Joint work with **Hiroyoshi Mitake, Takeshi Ohtsuka, and Hung V. Tran**.

We consider a level-set eikonal-curvature flow equation with an external force. Such a problem is considered as a model to describe an evolution of height of crystal surface by two-dimensional nucleation or possibly some class of growths by screw dislocations. For applications, it is important to estimate growth rate. Without an external source term the solution only spreads horizontally and does not grow vertically so the source term plays a key role for the growth.

Although the large time behavior of parabolic equations are well studied, the equations we study are degenerate parabolic equations where no diffusion effect exists in the normal to each level-set of a solution. Thus, very little is known even for growth rate. Our goal is to describe our recent progress on such type of problems. Ealier results are presented in the paper by H. Mitake, H.V. Tran and the lecturer published in SIAM Math. Anal. in 2016. A review paper is published in Proc. Int. Cong. of Math. in 2018.

In this talk, we first show the existence of asymptotic speed called growth rate. We also discuss asymptotic profile as well as estimates on growth rate.

THE TWO HYPERPLANE CONJECTURE

David Jerison
Massachusetts Institute of Technology, USA

I will introduce a conjecture that I call the *Two Hyperplane Conjecture*, saying that an isoperimetric surface that divides a convex body in half by volume is trapped between parallel hyperplanes. Emanuel Milman has shown that in its strongest, dimension-independent form, my conjecture implies the *Hyperplane Conjecture* of Kannan, Lovász and Simonovits in theoretical computer science, which says that the area of such an isoperimetric surface is comparable, by an absolute constant independent the convex body and its dimension, to the area of some hyperplane dividing the convex body in half. Their conjecture is closely related to several famous unsolved problems in high dimensional convex geometry. But unlike the hyperplane conjecture, the two-hyperplane conjecture has significance even in low dimensions.

I will relate the conjecture to qualitative and quantitative connectivity properties and regularity of area-minimizing surfaces, free boundaries and level sets of eigenfunctions, and report on work in progress with Guy David. The main theme of the talk is that the level sets of least energy solutions to scalar variational problems should be as simple as possible, but no simpler.

ON THE NONLINEAR STABILITY OF BLACK HOLES

Sergiu Klainerman
Princeton University, USA

Black holes are precise mathematical solutions of the Einstein field equations mainly represented by the famous two parameter Kerr family including, as a particular case, the Schwarzschild solution. To correspond to physical reality, i.e. to be more than mathematical artifacts, these solutions have to be stable under small perturbations. While there is today no doubt concerning the physical reality of black holes, based both on observational data and numerical simulations, an actual proof of stability remains a fundamental challenge of Mathematical and Geometric Analysis.

In my talk I will formulate the precise mathematical problem of the nonlinear stability of the Kerr family and describe the main results known so far. In the second part of the talk I will describe my recent result with J. Szeftel "Global Nonlinear Stability of Schwarzschild Spacetime under Polarized perturbations" - arXiv:1711.07597. The result establishes the full nonlinear stability of Schwarzschild spacetime under axially symmetric, polarized perturbations, i.e. stability of solutions of the Einstein vacuum equations for asymptotically flat $1+3$ dimensional Lorentzian metrics which admit a hyper-surface orthogonal space-like Killing vector-field with closed orbits. While building on the remarkable advances made in last 15 years on establishing quantitative linear stability, the paper introduces a series of new ideas among which we emphasize the *general covariant modulation* (GCM) procedure which allows us to construct, dynamically, the center of mass frame of the final state. The mass of the final state itself is tracked using the well known Hawking mass relative to a well adapted foliation itself connected to the center of mass frame. Our work here is the first to prove the nonlinear stability of Schwarzschild in a restricted class of nontrivial perturbations. To a large extent, the restriction to this class of perturbations is only needed to ensure that the final state of evolution is another Schwarzschild space.

ZERO SETS OF LAPLACE EIGENFUCNTIONS

Aleksandr Logunov
Princeton University, USA

We will discuss geometrical and analytic properties of zero sets of harmonic functions and eigenfunctions of the Laplace operator. For harmonic functions on the plane there is an interesting relation between local length of the zero set and the growth of harmonic functions. The larger the zero set is, the faster the growth of harmonic function should be and vice versa. Laplace eigenfunctions on two dimensional sphere are restrictions of homogeneous harmonic polynomials of three variables onto the sphere. Zero sets of such functions are unions of smooth curves with equiangular intersections. Topology of the zero set could be quite complicated, but the total length of the zero set of any spherical harmonic of degree n is comparable to n.

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EFFECTIVE BEHAVIOR OF RANDOM MEDIA

Felix Otto
Max Planck Institute for Mathematics in the Sciences, Germany

In engineering applications, heterogeneous media are often described in statistical terms. This partial knowledge is sufficient to determine the effective, i.e. large-scale behavior. This effective behavior may be inferred from the Representative Volume Element (RVE) method. I report on last years' progress on the quantitative understanding of what is called *stochastic homogenization of elliptic partial differential equations*: optimal error estimates of the RVE method, leading-order characterization of fluctuations, effective multipole expansions. Methods connect to elliptic regularity theory and to concentration of measure arguments.

IMPLICITLY CONSTITUTED FLUID FLOW MODELS: ANALYSIS AND APPROXIMATION

Endre Süli
University of Oxford, UK

Classical models describing the motion of Newtonian fluids, such as water, rely on the assumption that the Cauchy stress is a linear function of the symmetric part of the velocity gradient of the fluid. This assumption leads to the Navier-Stokes equations. It is known however that the framework of classical continuum mechanics, built upon a simple linear constitutive equation for the Cauchy stress, is too narrow to describe inelastic behavior of solid-like materials or viscoelastic properties of materials. Our starting point in this work is therefore a generalization of the classical framework of continuum mechanics, called the implicit constitutive theory, which was proposed recently in a series of papers by Rajagopal. The underlying principle of the implicit constitutive theory in the context of viscous flows is the following: instead of demanding that the Cauchy stress is an explicit (and, in particular, linear) function of the symmetric part of the velocity gradient, one may allow a nonlinear, implicit and not necessarily continuous relationship between these quantities. The resulting general theory therefore admits non-Newtonian fluid flow models with implicit and possibly discontinuous power-law-like rheology.

We develop the analysis of finite element approximations of implicit power-law-like models for viscous incompressible fluids. The Cauchy stress and the symmetric part of the velocity gradient in the class of models under consideration are related by a, possibly multi-valued graph. Using a variety of weak compactness techniques, we show that when the graph of the stress-strain relationship is maximal monotone a subsequence of the sequence of finite element solutions converges to a weak solution of the problem as the discretization parameter, measuring the granularity of the finite element triangulation, tends to zero. When the graph is nonmonotone, a subsequence of the sequence of finite element solutions is shown to converge to a gradient Young-measure solution of the problem. A key new technical tool in the analysis is a finite element counterpart of the Acerbi-Fusco Lipschitz truncation of Sobolev functions. The talk is based on a series of papers with Miroslav Bulíček and Josef Málek (Prague), Miles Caddick (Oxford), Lars Diening (Bielefeld), Christian Kreuzer (Dortmund), and ongoing research with Alexei Gazca-Orozco (Oxford) and Tabea Tscherpel (Aachen).

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GLOBAL ANALYSIS VIA MICROLOCAL TOOLS: FREDHOLM THEORY IN NON-ELLIPTIC SETTINGS

András Vasy
Stanford University, USA
Joint work with **Peter Hintz**.

In this lecture I will describe a microlocal framework for the Fredholm analysis of non-elliptic problems both on manifolds without boundary and manifolds with boundary, introduced in [8] and extended in [6] and various other works. Examples in which such a framework (or a similar framework) has recently been useful include wave propagation on black hole spacetimes, which is the key analytic ingredient for showing the stability of black holes [5, 4], analysis of the resolvent of the generator of the flow for dynamical systems [2], which is the key tool for the analysis of the Ruelle zeta function [1], Feynman propagators in quantum field theory [3, 9] and inverse problems, namely boundary rigidity and tensor tomography [7].

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THE VORTEX FILAMENT EQUATION, THE TALBOT EFFECT, AND NON-CIRCULAR JETS

Luis Vega
University of the Basque Country & BCAM, Spain
Joint work with **Valeria Banica and Francisco De La Hoz**.

We will propose the vortex filament equation as a possible toy model for turbulence, in particular because of its striking similarity to the dynamics of non-circular jets. This similarity implies the existence of some type of Talbot effect due to the interaction of non-linear waves that propagate along the filament. Another consequence of this interaction is the existence of a new class of multi-fractal sets that can be seen as a generalization of the graph of Riemann's non-differentiable function. Theoretical and numerical arguments about the transfer of energy will be also given.

POPULATION DYNAMICS AND CONTROL

Enrique Zuazua
DeustoTech, Bilbao, Basque Country & Universidad Autónoma de Madrid, Spain

Population dynamics is an old subject. Classical models in this field are written in terms of reaction-diffusion equations.

There is a wide literature concerning the dynamical properties of those systems. But much less is known from a control perspective. And control constitutes the ultimate proof of our understanding of a process.

This lecture will be devoted to present two recent results in this area. We first consider a bistable reaction-diffusion arising in the modelling of bilingual populations and then address models involving age structuring and spatial diffusion (of Lotka-McKendrick type).

As we shall see, both aspects require of an in depth understanding of the dynamics of the systems under consideration.

We shall present sharp results on our ability to steer the dynamics of those systems to a prescribed final configuration. Some open problems and future directions of research will also be presented.

This lecture is based on recent joint work in collaboration with D. Maity, C. Pouchol, E. Trélat, M. Tucsnak and J. Zhu.

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TALKS OF SESSION D21

ON THE RELAXED AREA OF THE GRAPH OF NONSMOOTH MAPS FROM THE PLANE TO THE PLANE

Giovanni Bellettini
University of Siena & ICTP, Italy
Joint work with **Alaa Elshorbagy, Maurizio Paolini, and Riccardo Scala**.

In various problems concerning area-minimizing surfaces, such as the non parametric Plateau problem, it is natural to have at hand a concept of area for graphs of nonsmooth scalar functions. In this scalar context, the correct notion to consider turns out to be the $L^1(\Omega)$-relaxation of the classical area functional $f \in \mathcal{C}^1(\Omega) \to \int_\Omega \sqrt{1+|\nabla f|^2}\, dx$; such a notion has been characterized and admits an integral representation in the space of functions of bounded variation in Ω [5, 2]. We are interested in a similar problem for vector-valued maps, more precisely for maps $u = (u_1, u_2) \to \mathbb{R}^2$ from a plane domain Ω to the plane. In this situation the classical area functional is polyconvex and, provided the map u is sufficiently smooth, say $u \in \mathcal{C}^1(\Omega; \mathbb{R}^2)$, reads as

$$A(u) := \int_\Omega \sqrt{1 + |\nabla u_1|^2 + |\nabla u_2|^2 + \Big(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\Big)^2}\, dxdy.$$

Its $L^1(\Omega;\mathbb{R}^2)$-relaxation $\overline{A}$ is the object of our interest (see also [4], [1]), in particular evaluated at discontinuous maps, the graphs of whice are therefore nonsmooth two-dimensional surfaces of codimension two. Assuming for simplicity Ω to be a disk centered at the origin in the source plane, we shall present some recent results [3] concerning piecewise constant maps $u_T : \Omega \to \{\alpha, \beta, \gamma\}$ taking three values and having a triple junction as a jump set; here

α, β, γ are three noncollinear vectors in the target plane. The appearence of solutions of certain Plateau-type problems, inducing a nonlocality phenomenon on $\overline{A}(u_T)$, will be pointed out.

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GLOBAL GRADIENT ESTIMATES FOR NONLINEAR ELLIPTIC PROBLEMS WITH NONSTANDARD GROWTH

Sun-Sig Byun
Seoul National University, South Korea

We are concerned with a quasilinear elliptic equation with non-standard growth condition over a non-smooth domain. The nonlinearity involves a non-uniformly ellipticity property and the boundary of the domain is sufficiently flat. We prove a global regularity estimate for the gradient of solutions in the frame of a generalized Sobolev space under substantially more general assumptions.

A BRIEF PERSPECTIVE ON TEMPERED DIFFUSION EQUATIONS

Juan Calvo
University of Granada, Spain
Joint work with **Antonio Marigonda and Giandomenico Orlandi**.

Tempered diffusion equations (also termed ”flux-saturated” or ”flux-limited” diffusion equations) are a class of non-linear versions of the standard diffusion equation displaying a mixture of parabolic and hyperbolic features. After reviewing the entropy solution framework and the qualitative properties of such equations, we will discuss the way in which those models fit in the variational framework developed by Jordan, Kinderlehrer and Otto to study evolution problems in terms of optimal transport theory. Then we will present some anisotropic versions of the foregoing theory and conclude with a tentative application to Developmental Biology: cytoneme networks.

THE SET OF TOPOLOGICAL SINGULARITIES OF VECTOR-VALUED MAPS

Giacomo Canevari
University of Verona, Italy
Joint work with **Giandomenico Orlandi**.

We introduce [1] an operator $\mathbf{S}$ on vector-valued maps $u\colon \Omega \subseteq \mathbb{R}^d \to \mathbb{R}^m$, which has the ability to capture the relevant topological information carried by u. In particular, this operator is defined on maps that take values in a closed submanifold $\mathcal{N}$ of the Euclidean space $\mathbb{R}^m$, and coincides with the distributional Jacobian in case $\mathcal{N}$ is a sphere. More precisely, the range of $\mathbf{S}$ is a set of maps whose values are flat chains with coefficients in a suitable normed abelian group. We use $\mathbf{S}$ to characterise strong limits of smooth, $\mathcal{N}$-valued maps with respect to Sobolev norms, extending a result by Pakzad and Rivière [2]. We present applications to the study of manifold-valued maps of bounded variation, and to the asymptotic behaviour of minimisers of Ginzburg-Landau type functionals, with $\mathcal{N}$-well potentials.

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REGULARITY AND RIGIDITY RESULTS FOR NONLOCAL MINIMAL GRAPHS

Matteo Cozzi

University of Bath, UK

Joint work with **Xavier Cabré, Alberto Farina, and Luca Lombardini**.

Nonlocal minimal surfaces are hypersurfaces of Euclidean space that minimize the fractional perimeter, a geometric functional introduced in 2010 by Caffarelli, Roquejoffre & Savin in connection with phase transition problems displaying long-range interactions.

In this talk, I will focus on the particular case of nonlocal minimal graphs and present some recent results obtained on their regularity and classification.

BERNOULLI FREE BOUNDARY PROBLEM FOR THE INFINITY LAPLACIAN

Graziano Crasta

Sapienza University of Rome, Italy

Joint work with **Ilaria Fragalà**.

We study the following interior Bernoulli free boundary problem for the infinity Laplacian:

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : \ u(x) > 0\}, \\ u = 1 & \text{on } \partial\Omega, \\ |\nabla u| = \lambda & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega \,, \end{cases}$$

where Ω is an open bounded connected domain in $\mathbb{R}^n$ ($n \geq 2$), and Δ_∞ is the infinity Laplace operator.

Our results cover existence, uniqueness, and characterization of viscosity solutions (for λ above a threshold representing the "infinity Bernoulli constant"), their regularity, and their relationship with the solutions to the interior Bernoulli problem for the p-Laplacian.

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SOLUTIONS TO TWO CONJECTURES IN BRANCHED TRANSPORT: STABILITY AND REGULARITY OF OPTIMAL PATHS

Antonio De Rosa
New York University, USA
Joint work with **Maria Colombo and Andrea Marchese**.

Models involving branched structures are employed to describe several supply-demand systems such as the structure of the nerves of a leaf, the system of roots of a tree and the nervous or cardiovascular systems. The transportation cost in these models is proportional to a concave power $\alpha \in (0, 1)$ of the intensity of the flow. We focus on the stability of the optimal transports, with respect to variations of the source and target measures. The stability was known when α is bigger than the critical threshold $1 - \frac{1}{n}$, where n is the dimension of the ambient space $\mathbb{R}^n$.

In [2, 3] we prove it holds for every exponent $\alpha \in (0,1)$ and we provide a counterexample for $\alpha = 0$. Thus we completely solve a conjecture of Bernot, Caselles and Morel, see [1, Problem 15.1]. Moreover, the robustness of our argument allows us to prove stability for more general lower semicontinuous cost functionals, called H-masses, introduced in [6] and also studied in [5]. Furthermore, in [4] we prove the stability for the mailing problem, which was completely open in the literature, solving a second conjecture in [1, Remark 6.13]. We use the latter result to show the regularity of the optimal networks, partially answering [1, Problem 15.5].

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ON TRACES AND SOME FINE PROPERTIES OF FUNCTIONS OF BOUNDED A-VARIATION

Franz Gmeineder
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Joint work with **Lars Diening**.

In this talk I address an interplay between trace theorems and selected fine properties of functions of bounded A-variation. In the case of BV-functions, the singular part of the gradients splits into the jump- and the Cantor part. We establish that, within the framework of functions of bounded A-variation, such a splitting requires refinement, and explain the connections to the corresponding trace theory.

DIMENSIONAL ESTIMATES AND RECTIFIABILITY FOR MEASURES SATISFYING LINEAR PDE CONSTRAINTS

Jonas Hirsch
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Joint work with **Adolfo Arroyo-Rabasa, Guido De Philippis, and Filip Rindler**.

We are going to present an rectifiability result for measures satisfying a linear PDE constraint. The presented rectifiability dimensions are optimal for many usual PDE operators, including all first-order systems and all second-order scalar operators. For instance it includes the the rectifiability results for functions of bounded variations (BV) and functions of bounded deformation (BD).

More precisely Let $\mathcal{A}$ be a k^{th}-order linear constant-coefficient PDE operator acting on $\mathbb{R}^m$-valued functions on $\mathbb{R}^d$ via

$$\mathcal{A}\varphi := \sum_{|\alpha|\leq k} A_\alpha \partial^\alpha \varphi \text{ for all } \varphi \in C^\infty(\mathbb{R}^d;\mathbb{R}^m),$$

where $A_\alpha \in \mathbb{R}^{n\times m}$ are (constant) matrices, $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N}\cup\{0\})^d$ is a multi-index and $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. We also assume that at least one A_α with $|\alpha| = k$ is non-zero. An $\mathbb{R}^m$-valued Radon measure $\mu \in \mathcal{M}(U;\mathbb{R}^m)$ defined on an open set $U \subset \mathbb{R}^d$ is said to be $\mathcal{A}$-*free* if

$$\mathcal{A}\mu = 0 \qquad \text{in the sense of distributions on } U. \tag{1}$$

Using the Lebesgue-Radon-Nikodým theorem we may define the *polar* of μ by

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) := \lim_{r\to 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}\,.$$

In the pioneering work [1], G. De Philippis and F. Rindler established a strong constraint on the direction on the polar on the singular part of an $\mathcal{A}$-free measure.

In this talk we are going to present a refinement of this pioniering result, the direction of the polar is further constrained on "lower dimensional parts" of the measure, [2].

As a consequence in the particular case of divergence-free tensors we are able to obtain refinements and new proofs of several known results on the rectifiability of varifolds and defect measures.

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THE GEOMETRY OF THE FREE BOUNDARY NEAR THE FIXED BOUNDARY GENERATED BY A FULLY NONLINEAR UNIFORMLY ELLIPTIC OPERATOR

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The dynamics of how the free boundary intersects the fixed boundary has been the object of study in the classical dam problem which is a mathematical model describing the filtration of water through a porous medium split into a wet and dry part. By localizing around a point at the intersection of free and fixed boundary, one is led to the following problem

$$\begin{cases} F(D^2u) = \chi_\Omega & \text{in } B_1^+ \\ u = 0 & \text{on } B_1' \end{cases}$$

where $\Omega = \big(\{u \neq 0\} \cup \{\nabla u \neq 0\}\big) \cap \{x_n > 0\} \subset \mathbb{R}^n_+$, $B_1' = \{x_n = 0\} \cap \overline{B_1^+}$, and F is a convex C^1 fully nonlinear uniformly elliptic operator. This talk focuses on the regularity problem for the free boundary $\Gamma = \partial\Omega \cap B_1^+$.

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CLASSICAL SOLUTION TO THE FRACTIONAL MEAN CURVATURE FLOW

Vesa Julin
University of Jyväskylä, Finland
Joint work with **Domenico La Manna**.

I will discuss about recent developments in the study of fractional mean curvature flow and introduce our recent work which is the first short time existence result of the smooth solution to the fractional mean curvature flow.

MAXIMAL SOLUTION OF THE LIOUVILLE EQUATION IN DOUBLY CONNECTED DOMAINS

Michał Kowalczyk
University of Chile, Chile
Joint work with **Angela Pistoia and Giusi Vaira**.

In this talk I will discuss a new existence result for the widely studied Liouville problem $\Delta u + \lambda^2 e^u = 0$ in a bounded, two dimensional, doubly connected domain with Dirichlet boundary conditions. I will show that for a sequence of $\lambda_n \to 0$ this equation has solutions that blow-up in in the whole domain. Profiles of the blowing-up solutions are related to a free boundary problem which gives a solution to an optimal partition problem for the given domain. I will also describe the role of the free boundary problem in other classical equations such as the mean field model or the prescribed Gaussian curvature equation.

SOBOLEV ESTIMATES FOR FIRST ORDER MEAN FIELD GAMES AND PLANNING PROBLEMS

Alpár R. Mészáros
University of California, Los Angeles, USA
Joint work with **Jameson Graber, Francisco Silva, and Daniela Tonon**.

In this talk we discuss Sobolev estimates for weak solutions of first order variational Mean Field Game systems (in the sense of Lasry-Lions) with coupling terms that are local functions of the density variable. Under some coercivity conditions on the coupling, we obtain first order Sobolev estimates for the density variable, while under similar coercivity conditions on the Hamiltonian we obtain second order Sobolev estimates for the value function. These results are valid both for stationary and time-dependent problems. In the latter case the estimates in the space variable are fully global in time, while the ones involving the time variable are local in time. In the same time we show how to obtain the same estimates for the mean field planning problem (introduced by P.-L. Lions). Our methods have their roots in Brenier's work on the regularity of the pressure field arising in weak solutions of the incompressible Euler equations (see [2]), which was improved later by Ambrosio-Figalli in [1]. The talk is based on the works [3, 4].

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ANISOTROPIC LIQUID DROP MODELS

Robin Neumayer
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Joint work with **Rustum Choksi and Ihsan Topaloglu** .

Anisotropic surface energies are a natural generalization of the perimeter functional that arise, for instance, in scaling limits for certain probabilistic models on lattices. Smoothness and ellipticity assumptions are sometimes imposed on the energy to improve analytic aspects of associated isoperimetric problems, but these assumptions are not always desirable for some applications nor checkable when the problem comes from a scaling limit. We consider an anisotropic variant of a model for atomic nuclei and show that minimizers behave in a fundamentally different way depending on whether or not the energy is smooth and elliptic. This is joint work with Choksi and Topaloglu.

ENERGY MINIMIZING MAPS INTO FINSLER MANIFOLDS AND OPTIMAL ONE-DIMENSIONAL NETWORKS

Giandomenico Orlandi
University of Verona, Italy
Joint work with **Sisto Baldo and Annalisa Massaccesi**.

We consider certain natural variational problems for maps valued into manifolds equipped with a Finsler norm, study their relaxation and show the emergence of optimal one dimensional

networks where energy concentrates, providing a link with the classical Steiner Tree problem or, more generally, with Gilbert-Steiner irrigation-type problems.

THEME & VARIATIONS ON $\operatorname{div} \mu = \sigma$

Filip Rindler
University of Warwick, UK
Joint work with **A. Arroyo-Rabasa, G. De Philippis, J. Hirsch, and A. Marchese**.

The PDE $\operatorname{div} \mu = \sigma$ for (vector) measures μ and σ appears - sometimes in a slightly hidden way - in many different problems of geometric measure theory and the calculus of variations, for instance in the structure theory of normal currents, Lipschitz functions and varifolds. In this talk I will survey a number of recent results about this equation and other related PDEs. As applications, I will discuss the structure of singularities of solutions, dimensional estimates, and several versions of Rademacher's theorem (both in Euclidean and non-Euclidean settings).

THE CAHN-HILLIARD EQUATION: EXISTENCE RESULTS AND QUALITATIVE PROPERTIES

Matteo Rizzi
University of Chile, Chile
Joint work with **Michal Kowalczyk**.

In the talk I will present the construction of a family $\{u_\varepsilon\}$ of solutions to the Cahn-Hilliard equation

$$-\varepsilon \Delta u_\varepsilon = \varepsilon^{-1}(u_\varepsilon - u_\varepsilon^3) - \ell_\varepsilon, \qquad \ell_\varepsilon \in \mathbb{R},$$

whose zero level set is prescribed and approaches, as $\varepsilon \to 0$, a given complete, embedded, k-ended constant mean curvature surface. Moreover, I will present some classification results, dealing with properties such as boundedness, monotonicity and radial symmetry.

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OBSTACLE PROBLEMS FOR BV MINIMIZERS, AND WEAK SUPERSOLUTIONS TO THE 1-LAPLACE EQUATION

Thomas Schmidt
Universität Hamburg, Germany
Joint work with **Lisa Beck and Christoph Scheven**.

The talk will be concerned with Dirichlet and obstacle problems for the total variation, the area integral, and possibly more general variational integrals on the space of functions of bounded variation. In particular, it is planned to discuss duality-based connections to (super)solutions of PDEs of 1-Laplace and minimal surface type.

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RECENT ADVANCES IN MULTI-MARGINAL OT APPLIED TO DFT

Federico Stra
Ecole Polytechnique Fédérale de Lausanne, Switzerland
Joint work with **M. Colombo and S. Di Marino**.

Multi-marginal optimal transport has been adopted to approximate the electron-electron interaction energy in the context of Density Functional Theory (DFT).

In this talk I will review the OT formulation of the DFT problem and mention what conditions ensure the finiteness and the continuity of multi-marginal optimal transport with repulsive cost (expressed in terms of a suitable concentration property of the measure).

Finally I will present some recent results regarding the Γ-convergence of the functionals for the study of the semiclassical limit of ground states.

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ON THE LOGARITHMIC EPIPERIMETRIC INEQUALITY

Bozhidar Velichkov
Université Grenoble-Alpes, France
Joint work with **Maria Colombo and Luca Spolaor**.

This talk is dedicted to some recent advances on the regularity of the free boundaries arising in variational minimization problems. In particular, we will present a new variational approach for the analysis of singularities: the *logarithmic epiperimetric inequalities*, which was already applied to several different free boundary problems: the obstacle problem [2], the thin-obstacle problem [3, 7], the one-phase Alt-Caffarelli problem [4], to area-minimizing currents [5], and to parabolic free boundary problems [8].

The focus of this talk is on the classical obstacle problem:

$$\min\Big\{\int_{B_1}\big(|\nabla u|^2+u\big)\,dx\ :\ u\in H^1(B_1),\ u\geq 0 \text{ in } B_1,\ u \text{ is prescribed on } \partial B_1\Big\}. \tag{OB}$$

It is well-known that, given a solution u and setting $\Omega_u := \{u>0\}$, the free boundary $\partial\Omega_u\cap B_1$ can be decomposed into a regular part, $Reg\,(\partial\Omega_u)$, and a singular part, $Sing\,(\partial\Omega_u)$, where

- $Reg\,(\partial\Omega_u)$ is a smooth manifold (this result was proved by Caffarelli in [1]);
- $Sing\,(\partial\Omega_u)$ are the points, at which the Lebesgue density of the set $\{u=0\}$ vanishes.

In this talk, we will prove a *logarithmic epiperimetric inequality* for the Weiss' boundary adjusted energy, from which we will deduce that the singular set $Sing\,(\partial\Omega_u)$ is $C^{1,\log}$-regular, that is, it is contained into the countable union of $C^{1,\log}$-regular manifolds. This results was first proved in [2]. Recently, Figalli and Serra [6] showed that this result is also optimal in the sense that the logarithmic modulus of continuity cannot be improved for general singularities.

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RENORMALIZED SOLUTIONS FOR PARABOLIC PROBLEMS WITH ANISOTROPIC STRUCTURE AND NONSTANDARD GROWTH

Anna Zatorska-Goldstein
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Joint work with **Iwona Chlebicka and Piotr Gwiazda**.

We investigate nonlinear parabolic problems where the ellipticity and the growth conditions for the leading part of the operator is driven by an inhomogeneous and anisotropic function of the Orlicz type. We establish the existence and the uniqueness of renormalized solutions when merely integrable data are allowed. Fully anisotropic, non-reflexive Orlicz-Sobolev spaces provide a natural functional framework associated with these problems. Our results cover in particular the case of variable exponent growth, with the exponent depending both on time and space variables.

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